

Homogeneous Functions

Consider a function involving two independent variables x and y
 $Z = f(x, y)$

① Suppose, $Z = f(x, y) = \frac{x^2}{y^2} + \frac{y^2}{x^2}$

$$f(kx, ky) = \frac{(kx)^2}{(ky)^2} + \frac{(ky)^2}{(kx)^2}$$

$$= \frac{k^2 x^2}{k^2 y^2} + \frac{k^2 y^2}{k^2 x^2}$$

$$= \frac{x^2}{y^2} + \frac{y^2}{x^2}$$

$$= k^0 \cdot f(x, y)$$

$$= f(x, y) = Z$$

② Suppose, $Z = f(x, y) = x^3 + 3x^2y + 3xy^2 + y^3$

$$f(kx, ky) = (kx)^3 + 3(kx)^2(ky) + 3(kx)(ky)^2 + (ky)^3$$

$$= k^3 x^3 + 3k^3 x^2 y + 3k^3 x y^2 + k^3 y^3$$

$$= k^3 (x^3 + 3x^2 y + 3x y^2 + y^3)$$

$$= k^3 f(x, y)$$

$$= k^3 (Z)$$

⇒ when inputs are changed by the same proportion k , the function's Z changes by some power of k . k times change in x and y has changed the function's Z by k^0, k^3 times respectively. This property renders these functions to be homogeneous functions.

All functions are not homogeneous.

$$z = x^2 + 2y$$

$$\begin{aligned} z &= k^2 x^2 + 2ky \\ &= k(kx^2 + 2y) \\ &\neq kz. \end{aligned}$$

→ It is only when the degree of homogeneity is one that the function is referred to as linearly homogeneous function.

→ A function can be linearly homogeneous yet may not be linear in relation.

$$z = f(x, y) = \sqrt{x^2 + 3xy + y^2}$$

The word "linear" here should always be interpreted only as a synonym of the first degree."

Properties of Linear Homogeneous Functions

1 - The given linear homogeneous function can be written in either of the following two forms:

$$z = x \cdot \phi\left(\frac{y}{x}\right) \quad \text{or} \quad z = y \cdot \psi\left(\frac{x}{y}\right)$$

$$\text{Let } k = \frac{1}{x} \quad z = f(xy)$$

$$f(kx, ky) = k \cdot f(x, y)$$

$$\left(1, \frac{y}{x}\right) = \frac{1}{x} \cdot f(x, y)$$

$$\begin{aligned} f(x, y) &= x \cdot \left(1, \frac{y}{x}\right) \\ &= x \cdot \phi\left(\frac{y}{x}\right) \end{aligned}$$

$$\text{if } k = \frac{1}{y}$$

$$f\left(\frac{x}{y}, 1\right) = \frac{1}{y} \cdot f(x, y)$$

$$f(x, y) = y \cdot \psi\left(\frac{x}{y}\right)$$

2- The partial derivatives of the dependent variable of the given linear homogeneous function (i.e. $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$) are functions of the ratio of x to y .

From property I, $z = x \phi\left(\frac{y}{x}\right)$

$$\frac{\partial z}{\partial x} = \phi\left(\frac{y}{x}\right) + x \cdot \phi'\left(\frac{y}{x}\right) \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) \quad \text{Product Rule}$$

$$\frac{\partial z}{\partial x} = \phi\left(\frac{y}{x}\right) - \frac{y}{x} \phi'\left(\frac{y}{x}\right)$$

\therefore partial derivative of z with respect to x is function of ratio of the independent variables.

Hence, $\frac{\partial z}{\partial x}$ is homogeneous of degree 0 in y and x .

3- The linearly homogeneous functions satisfy "Euler's Theorem".

$$z = f(x, y)$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = z$$

Substitute values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ obtained in Property 2 in

$$\left[x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} \right]$$

$$= x \left\{ \phi\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right) \phi'\left(\frac{y}{x}\right) \right\} + y \phi'\left(\frac{y}{x}\right)$$

$$= x \cdot \phi\left(\frac{y}{x}\right) - y \cdot \phi'\left(\frac{y}{x}\right) + y \cdot \phi'\left(\frac{y}{x}\right)$$

$$= x \cdot \phi\left(\frac{y}{x}\right)$$

$$= f(x, y) = z \quad (\text{Property 1})$$

Example: $Z = \sqrt[3]{x^2 y}$

$$\frac{\partial Z}{\partial x} = \frac{2}{3} x^{-1/3} x^{2/3} y^{1/3} = \frac{2}{3} x^{1/3} y^{1/3}$$

$$\frac{\partial Z}{\partial y} = \frac{1}{3} x^{2/3} y^{-2/3}$$

Therefore $x \cdot \frac{\partial Z}{\partial x} + y \cdot \frac{\partial Z}{\partial y} = x \left(\frac{2}{3} x^{1/3} y^{1/3} \right) + y \left(\frac{1}{3} x^{2/3} y^{-2/3} \right)$

$$= \frac{2}{3} x^{4/3} y^{1/3} + \frac{1}{3} x^{2/3} y^{1/3}$$

$$= \frac{2}{3} x^{2/3} y^{1/3} x + \frac{1}{3} x^{2/3} y^{1/3} y$$

$$= \frac{2}{3} x^{2/3} y^{1/3} (x + y) = \frac{2}{3} x^{2/3} y^{1/3} (3x^{2/3} y^{1/3}) = 2x^{2/3} y^{1/3} = Z$$

For Cobb-Douglas Function

$$Q = AK^a L^{1-a}$$

where, K is capital, L is labour and a is constant

$$K \frac{\partial Q}{\partial K} + L \frac{\partial Q}{\partial L} = Q$$

$$\frac{\partial Q}{\partial K} = a^2 K^{a-1} L^{1-a}$$

$$\frac{\partial Q}{\partial L} = a(1-a) K^a L^{-a}$$

Therefore, $K \frac{\partial Q}{\partial K} + L \frac{\partial Q}{\partial L} = K (a^2 K^{a-1} L^{1-a}) + L [a(1-a) K^a L^{-a}]$

$$= a^2 K^a L^{1-a} + a(1-a) K^a L^{1-a}$$

$$= a K^a L^{1-a} [a + (1-a)]$$

$$= a K^a L^{1-a} = Q \text{ hence proved.}$$

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$$\left[x_2 = x_1 \frac{\partial c}{\partial x_1} \right] \text{ and } c = e + c \left[\frac{\partial c}{\partial x_1} \right]$$

Harris-Todaro Model of Migration and Unemployment

The Harris-Todaro (H-T) model is based on the experiences of tropical Africa facing the problems of rural-urban migration and urban unemployment. The labour migration is due to rural-urban differences in average expected wages. To remove urban unemployment, Harris and Todaro suggest a subsidised minimum wage through a lumpsum tax.

Assumptions :

- 1 - There are two sectors in the economy; the rural or Agricultural sector (A) and the urban or manufacturing sector (M).
2. The rural sector produces X_A units of agricultural goods and the urban sector produces X_M units of manufactured goods. Each sector produces only one unit.

- 3 - The model operates in the short run.
- 4 - Capital is available in fixed quantities (K) in the two sectors.
- 5 - There are N workers in economy with N_A and N_U numbers employed in the rural and urban sector respectively.
- 6 - The number of urban jobs available (N_U) is exogenously fixed. In the rural sector some work is always available. Therefore, the total urban labour force comprises $N - N_A$ along with an available supply of rural migrants.
- 7 - The urban wage is fixed at w_U and the rural wage at w_A , $w_U > w_A$.
- 8 - The rural wage equals the rural marginal product of labour and the urban wage is exogenously determined.
- 9 - Rural-urban migration continues so long as the expected urban real income is more than the real agricultural income.
- 10 - The expected urban real income is equal to the proportion of urban labour force actually employed multiplied by the fixed maximum urban wage.
- 11 - There is perfect competition among producers in both the sectors.
- 12 - The price of the agricultural goods is determined directly by the relative quantity of the two goods produced in both the sectors.

Model

Give the assumptions, Harris-Todaro explains their model mathematically.

Output in rural sector is function of labour

$$X_A = f(N_A, \bar{L}, K_A) \quad f' > 0; f'' < 0 \quad \text{--- (1)}$$

where,

X_A → output of agricultural goods

N_A → Rural labour units employed to produce this output.

\bar{L} → fixed given labour.

K_A → Fixed quantity of available capital in the rural sector.

Similarly, output in the urban sector

$$X_M = f(N_M, K_M) \quad f' > 0; f'' < 0 \quad \text{--- (2)}$$

where →

X_M → Output of Manufactured goods

N_M → Urban labour units

K_M → fixed quantity of available capital in the urban sector.

Total labour available in the economy is N .

$$N_A + N_M \leq N \quad N_A, N_M > 0$$

price determination

$$P = P\left(\frac{X_M}{X_A}\right) \quad P' > 0$$

where →

P → price of agricultural goods in terms of the price of manufactured goods which is a function of the relative output of agricultural and manufactured goods.

The agricultural wage equals the value of marginal product (MP) of labour expressed in terms of the manufactured goods,

$$w_a = f'_a(N_a) = P(f'_M) \quad \text{--- (4)}$$

In the urban sector, the producers are wage-takers and they aim at profit maximisation.

$$w_u = f'_M(N_u)$$

However, in this economy, the urban real minimum wage (\bar{w}_u) is at a lower level due to institutional or political factors.

$$\bar{w}_u = f'_M \gg w_u \quad \text{--- (5)}$$

Wage in the urban sector is equal to MP of labour because of the price-taking behaviour of producers. This assumption is called the wage-rigidity axiom.

Assuming wage to be flexible, if wages are above \bar{w}_u , there will be an excess supply of labour in the urban sector and competition among producers will drive w_u to the level \bar{w}_u . Thus profit maximisation condition becomes

$$\bar{w}_u = f'_M(N_u)$$

The urban expected wage which leads to the migration of workers from the rural to the urban sector is

$$w_u^e = \bar{w}_u \cdot \frac{N_u}{N_u}, \quad \frac{N_u}{N_u} \leq 1 \quad \text{--- (6)}$$

where,

the expected real wage (w_u^e) in the urban sector is equal to the urban real minimum wage (w_u) adjusted for the proportion of the total urban labour force (N_u) actually employed, i.e. $N_u/N_u = 1$, there is full employment in the urban sector and the expected real wage equals the real minimum wage, i.e. $w_u^e = w_u$.

The total labour endowment (\bar{N}) in the economy is $\bar{N} = \bar{N}_A + \bar{N}_B = N_A + N_B$. — (7)

The equilibrium condition is

$$w_A = w_u^e$$

This is based on the hypothesis that migration from the rural to the urban sector is positive function of urban-rural wage differential.

$$N_B^* = f\left(w_u \cdot \frac{N_u}{N_B} p f'\right) \quad f' > 0; f(0) = 0 \quad \text{--- (8)}$$

This implies that migration from the rural to urban sector will cease when the expected wage differential is zero i.e. $w_A = w_u^e$.

This completes the description of an H-T economy. But the above condition does not ensure equilibrium in the entire economy. This requires satisfying equations from (1) to (8). The H-T model contains eight equations and eight unknowns: $X_A, X_B, N_A, N_B, w_A, w_u, N$ and p .